Exercise 75

Find all points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

Solution

Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^2y^2 + xy) = \frac{d}{dx}(2)$$

Use the chain rule to differentiate $y = y(x)$.

$$\frac{d}{dx}(x^2y^2) + \frac{d}{dx}(xy) = 0$$

$$\left[\frac{d}{dx}(x^2)\right]y^2 + x^2\left[\frac{d}{dx}(y^2)\right] + \left[\frac{d}{dx}(x)\right]y + x\left[\frac{d}{dx}(y)\right] = 0$$

$$(2x)y^2 + x^2(2y)\frac{dy}{dx} + (1)y + x\frac{dy}{dx} = 0$$

Set dy/dx to -1 and simplify the equation.

$$2xy^2 + 2x^2y(-1) + y + x(-1) = 0$$

$$2xy^2 - 2x^2y + y - x = 0$$

Multiply both sides by xy.

$$2x^2y^3 - 2x^3y^2 + xy^2 - x^2y = 0$$

$$y(2x^2y^2 + xy) - x(2x^2y^2 + xy) = 0$$

$$(y - x)(2x^2y^2 + xy) = 0$$

$$(y - x)[x^2y^2 + (x^2y^2 + xy)] = 0$$

$$(y - x)(x^2y^2 + 2) = 0$$

Since x^2y^2 is never negative, the factor $x^2y^2 + 2$ is positive. Divide both sides by $x^2y^2 + 2$.

$$y - x = 0$$

$$y = x$$

Find where this line intersects the curve by solving the following system of equations.

$$\left.\begin{aligned} y &= x \\ x^2y^2 + xy &= 2 \end{aligned}\right\}$$

Substitute the formula for y into the second formula and solve for x.

$$x^2(x)^2 + x(x) = 2$$

$$x^4 + x^2 = 2$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2 + 2)(x^2 - 1) = 0$$

$$(x^2 + 2)(x + 1)(x - 1) = 0$$

$$x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = -1 \quad \text{or} \quad x = 1$$

The y-value corresponding with $x = -1$ is $y = -1$, and the y-value corresponding with $x = 1$ is $y = 1$. Therefore, the points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1 are $(-1, -1)$ and $(1, 1)$.

The graph below illustrates the curve and the tangent lines with slope −1.